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A Geometrical Analysis of the Efficient Outcome Set in Multiple Objective Convex Programs with Linear Criterion Functions

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Abstract. This article performs a geometrical analysis of the efficient outcome set Y_E of a multiple objective convex program (MLC) with linear criterion functions. The analysis elucidates the facial structure of Y_E and of its pre-image, the efficient decision set X_E. The results show that Y_E often has a significantly-simpler structure than X_E. For instance, although both sets are generally nonconvex and their maximal efficient faces are always in one-to-one correspondence, large numbers of extreme points and faces in X_E can map into non-facial subsets of faces in Y_E, but not vice versa. Simple tests for the efficiency of faces in the decision and outcome sets are derived, and certain types of faces in the decision set are studied that are immune to a common phenomenon called "collapsing." The results seem to indicate that significant computational benefits may potentially be derived if algorithms for problem (MLC) were to work directly with the outcome set of the problem to find points and faces of Y_E, rather than with the decision set.

Key words: Multiple objective mathematical programming, global optimization, efficient points, faces, convex geometry.

1. Introduction

The multiple objective mathematical programming problem involves the simultaneous maximization of $p \geqslant 2$ noncomparable criterion functions over a nonempty set. The concept of an efficient solution has played a useful role in the analysis and solution of this problem. In particular, many of the approaches for analyzing and solving this problem generate either all or at least some of the efficient solution set. In this way, inherent tradeoffs in the problem are revealed, and most-preferred solutions can be sought. Included among these types of approaches, for instance, are the vector maximization approach, interactive approaches, and several others (see, for instance, the books and general surveys by Cohon [10], Evans [21], Goicoechea *et al.* [24], Kuhn and Tucker [29], Luc [32], Ringuest [36], Rosenthal [38], Sawaragi *et al.* [39], Stadler [40], Steuer [42], Yu [48, 49], Zeleny [51] and references therein).

Adopting the notation of Geoffrion [23], we may represent a multiple objective mathematical programming problem (M) by

$$\text{VMAX: } f(x), \quad \text{subject to } x \in X,$$

where

$$f(x) = [f_1(x), f_2(x), \ldots, f_p(x)],$$

X is a nonempty set in $\mathbb{R}^n$, and, for each $j \in \{1, 2, \ldots, p\}$, $f_j : X \to \mathbb{R}$. A point $\overline{x}$ is said to be an *efficient solution* (or *decision*) for problem (M) when $\overline{x} \in X$ and there exists no point $x \in X$ such that $f(x) \geqslant f(\overline{x})$ and $f(x) \neq f(\overline{x})$. The set X is called the *decision set* for problem (M), and the set X_E of all efficient decisions is called the *efficient* (or *nondominated*) *set* for problem (M). In this article, we will also often refer to X_E as the *efficient decision set*. Problem (M) is said to be a *multiple objective convex (linear) programming problem* when each objective function f_j is concave (linear) and the decision set X is convex (polyhedral).

Let Y denote the set $f(X) = \{f(x) | x \in X\}$, and let

$$Y^{\leqslant} = \{y \in \mathbb{R}^p | y \leqslant f(x) \text{ for some } x \in X\}.$$

The set $Y = f(X)$ is called the *outcome set* (or *image*) *of X under f*. The set $Y^{\leqslant}$ and generalizations of this set have been used to good effect by many researchers, including, for instance, Benson [4], Bitran and Magnanti [8], Dauer and Saleh [17], and Yu [47]. Both the outcome set Y and the set $Y^{\leqslant}$ take the natural viewpoint of considering f to be a mapping from $\mathbb{R}^n$ into $\mathbb{R}^p$. In particular, in outcome space, if we define the efficient sets Y_E and $Y_E^{\leqslant}$ by

$$Y_E = \{\overline{y} \in Y | \text{there is no } y \in Y \text{ such that } y \geqslant \overline{y} \text{ and } y \neq \overline{y}\} \tag{1}$$

and

$$Y_E^{\leqslant} = \{\overline{y} \in Y^{\leqslant} | \text{there is no } y \in Y^{\leqslant} \text{ such that } y \geqslant \overline{y} \text{ and } y \neq \overline{y}\},$$

respectively, then X_E is the pre-image under f of $Y_E = Y_E^{\leqslant}$. We will refer to $Y_E = Y_E^{\leqslant}$ as the *efficient outcome set* for problem (M).

Mathematically, the task of generating X_E or Y_E or significant portions of these sets for problem (M) is a difficult global optimization problem. This is because X_E and Y_E are, in general, nonconvex sets, even in the case of multiple objective linear programming. Confounding the situation further is the fact that neither X_E nor Y_E is given in the traditional mathematical programming format as a system of functional inequalities.

Largely because of the need for methods for generating X_E or portions of X_E, researchers have been studying the mathematical structure of X_E for many years. During the course of this study, these researchers realized that many of the key mathematical properties of X_E can be derived and understood more easily by focusing on the outcome set Y (or on $Y^{\leqslant}$) and on the efficient outcome set Y_E instead of on X and X_E. Numerous instances of the beneficial use of this approach can be cited. For example it has been used in studies concerning the existence of points in X_E [4–5, 8–9, 11, 19, 25, 28, 44, 48], the "domination property" of X_E [6, 26, 31, 44], the connectedness of X_E [8, 30, 33, 45], the density of the "properly" efficient points in X_E [2, 8, 27, 46–47], the stability of X_E [34, 39, 41, 43], and the contractibility of X_E [32].

In recent years, some notable computational progress has been made, in particular, for the multiple objective linear programming problem (MLL) by using the outcome set approach. Dauer [13] and Dauer and Liu [15], for example, have devised a simplex-like procedure for problem (MLL) that generates all of the extreme points and edges of Y_E rather than those of X_E. Since in problem (MLL), Y_E usually has a significantly-simpler structure with many fewer extreme points and edges than X_E, these procedures offer great practical promise. For example, Dauer and Liu [15] have estimated for a certain water resources planning model that their method might require only slightly more than one-eighth of the computational effort of a typical decision set-based method for generating all of the efficient extreme points of X.

These first outcome set-based computational procedures were to a significant extent made possible by some fundamental geometrical results that Dauer [12] had derived earlier for multiple objective linear programs. Motivated by a desire to avoid the large cost of describing the efficient set in decision space, Dauer derived several geometrical properties of the efficient set in outcome space. For instance, he showed when and by how much the dimension of a face of X_E is reduced when the face is mapped into Y_E by problem (MLL). He gave sufficient conditions for this reduction to be zero, and he showed how faces of X_E can "collapse" in Y_E into non-facial subsets of Y_E. In these and similar ways, he helped to clarify how the linear mapping of X_E onto Y_E given in problem (MLL) can considerably simplify the structure of the efficient set, reduce its dimension, and decrease the number of extreme points and faces that it contains.

In this article we will perform a geometrical analysis of the efficient outcome set of a multiple objective convex programming problem (MLC) with linear objective functions. This analysis is to a great extent motivated by a desire to provide some foundations for developing procedures for problem (MLC) that generate points in Y_E rather than in X_E.

The analysis will show that for problem (MLC), as in the linear case, the efficient outcome set often has a significantly-simpler structure than the efficient set in decision space. For example, we will show that in many cases of problem (MLC), the efficient decision set must contain at least as many extreme points as the efficient outcome set. More generally, we will show that although the maximal efficient faces of the decision and outcome sets in problem (MLC) are always in one-to-one correspondence, large numbers of non-maximal efficient faces in decision space can map into non-facial subsets of the efficient outcome set, but not vice-versa. Furthermore, we will show that the dimensions of the efficient faces in the decision set always exceed or equal the dimensions of their images in the efficient outcome set, even when these images are not faces of the outcome set.

On the other hand, the analysis will also show that in certain special cases of problem (MLC), the efficient decision and outcome sets can sometimes display characteristics that partially contradict the notion that, of the two sets, the efficient outcome set has the simpler structure. In particular, we will show for some cases of problem (MLC), including for some multiple objective linear programs, that the number of extreme points in the efficient decision set can actually be *smaller* than the number of extreme points in the efficient outcome set.

By focusing on $Y^{\leqslant}$ and on $Y_E^{\leqslant}$, many (but not all) of the results in this article can be extended to the more general multiple objective convex program (MCC) with concave criteria and a convex decision set. However, we shall not stop here to present these extensions.

The organization of this article is as follows. In Section 2 we give some notation and preliminaries. Section 3 elucidates the fundamental facial structure of the efficient outcome set of problem (MLC). It shows that the efficient outcome set is a union of relative interiors of faces of the outcome set. A similar result is shown to hold for the efficient decision set. The section also shows the relationships that can exist between the numbers of efficient extreme points in the decision set and the outcome set and between the numbers and dimensions of the efficient faces in these two sets. In Section 4 we derive simple tests for the efficiency of faces in the decision and outcome sets, and we show that the sets of maximal efficient faces in decision and outcome space are in one-to-one correspondence. We also analyze a phenomenon called *collapsing* [12] in this section. Some conclusions and suggestions for further research are given in Section 5. Examples form a fundamental part of the analysis and are therefore given throughout the article.

2. Notation and Preliminaries

Assume henceforth that each criterion function f_j in problem (M) is a linear function defined on $\mathbb{R}^n$ and that X is convex as well as nonempty in $\mathbb{R}^n$. In this case, problem (M) is a multiple objective convex programming problem with linear objective functions. We will denote this problem as problem (MLC). In contrast to multiple objective linear programs, both the decision set X and the outcome set $Y = f(X)$ of problem (MLC) are, in general, nonpolyhedral sets.

We will be using the following notations and terms throughout the remainder of the article. Assume that $q \geqslant 1$ is an integer.

For any real numbers α and β satisfying $\alpha < \beta$, the sets given by $\{x \in \mathbb{R} | \alpha \leqslant x \leqslant \beta\}$ and $\{x \in \mathbb{R} | \alpha < x < \beta\}$ will be denoted $[\alpha, \beta]$ and (α, β), respectively.

For any set $U \subseteq \mathbb{R}^q$, (aff U) and (sub U) will denote the affine hull of U and the subspace generated by U, respectively. If, in addition, $V \subseteq \mathbb{R}^q$, then the difference $U \backslash V$ will denote $\{x \in \mathbb{R}^q | x \in U, x \notin V\}$.

For any convex set $U \subseteq \mathbb{R}^q$, the relative interior and the relative boundary of U will be denoted (ri U) and (rb U), respectively. Recall that if, in addition, U is nonempty, then a point $u^\circ \in U$ is called an *extreme point* of U if $u^\circ = \lambda u^1 + (1-\lambda)u^2$ for some $u^1, u^2 \in U$ and some $\lambda \in (0,1)$ implies that $u^1 = u^2 = u^\circ$. More generally, a *face* of U is a convex subset H of U such that for any closed line segment $L = \{x \in \mathbb{R}^q | x = \lambda x^1 + (1-\lambda)x^2$ for some $\lambda \in [0,1]\}$ connecting two points x^1 and x^2 of U that satisfies $(ri\ L) \cap H \neq \emptyset$, both x^1 and x^2 must lie in H. Additionally, the *dimension* of U is the dimension of (aff U). We will denote the set of all extreme points and the dimension of a nonempty convex set U in R^q by U_{ex} and dim U, respectively.

Let D be a $p \times n$ matrix of real numbers. Without loss of generality, we will represent problem (MLC) henceforth as

$$\text{VMAX} : Dx, \quad \text{subject to } x \in X,$$

where the rows of D contain the coefficients of the p linear objective functions of the problem. The outcome set Y for problem (MLC) is then given by $Y = D[X]$, where for any set $Z \in \mathbb{R}^n$, $D[Z]$ denotes $\{y \in \mathbb{R}^p | y = Dz$ for some $z \in Z\}$. From Rockafellar [37], Y is a nonempty convex set in $\mathbb{R}^p$. Our main goal is to geometrically analyze the efficient outcome set Y_E of problem (MLC), where Y_E is given by (1). As part of this analysis, since $Y_E = D[X_E]$, we will also study the geometry of the efficient decision set X_E of problem (MLC) and, in particular, the effects of the mapping D on X_E.

To conclude this section, we present a result that will be used throughout the article. The result confirms that $Y_E = D[X_E]$ and can be easily proven from the definitions.

PROPOSITION 2.1.
(a) *For any $y^\circ \in Y_E$, if $x^\circ \in X$ satisfies $Dx^\circ = y^\circ$, then $x^\circ \in X_E$.*
(b) *For any $x^\circ \in X_E$, if $y^\circ = Dx^\circ$, then $y^\circ \in Y_E$.*

3. Fundamental Facial Structure

Since $Y = D[X]$ is a nonempty convex set in $\mathbb{R}^p$, we know from convex analysis that the collection T of all relative interiors of nonempty faces of Y is a partition of Y, i.e., the sets in T are disjoint and their union is Y (cf., e.g., Rockafellar [37]). Our first result will show that although $Y_E \subseteq Y$ is not generally a convex set, a similar partition T_E for Y_E exists. The key to deriving this result comes from the following theorem.

THEOREM 3.1. *Let G be a face of Y. If $(ri\ G)$ contains a point $y^\circ \in Y_E$, then $G \subseteq Y_E$.*

Proof. Let $y^1 \in G$. If $y^1 = y^\circ$, then $y^1 \in Y_E$. Otherwise, since $y^\circ \in (ri\ G)$, we may choose a scalar $\theta > 1$ such that $y^1 + \theta(y^\circ - y^1) \in G$ [37, p. 47]. Let $\overline{y} = y^1 + \theta(y^\circ - y^1)$. Then $y^\circ = (1 - \frac{1}{\theta})y^1 + \frac{1}{\theta}\overline{y}$. Since $\theta > 1$, this implies that y° is a strict convex combination of y^1 and $\overline{y}$. Assume that $\hat{y} \in Y$ satisfies $\hat{y} \geqslant y^1$. Let y be defined by

$$y = \left(1 - \frac{1}{\theta}\right)\hat{y} + \frac{1}{\theta}\,\overline{y}. \tag{2}$$

Then y is a strict convex combination of $\hat{y} \in Y$ and $\overline{y} \in G \subseteq Y$, so that $y \in Y$. Furthermore, since $\hat{y} \geqslant y^1$ and $(1 - \frac{1}{\theta}) > 0$, $(1 - \frac{1}{\theta})\hat{y} + \frac{1}{\theta}\overline{y} \geqslant (1 - \frac{1}{\theta})y^1 + \frac{1}{\theta}\overline{y}$. From (2) and the expression for y° derived earlier, this inequality states that $y \geqslant y^\circ$. Since $y^\circ \in Y_E$ and $y \in Y$, $y = y^\circ$ must hold. It follows that

$$\left(1 - \frac{1}{\theta}\right)\hat{y} + \frac{1}{\theta}\,\overline{y} = \left(1 - \frac{1}{\theta}\right)y^1 + \frac{1}{\theta}\,\overline{y}.$$

Since $(1 - \frac{1}{\theta}) > 0$, this equality implies that $\hat{y} = y^1$. Because $\hat{y}$ was assumed to be any element of Y for which $\hat{y} \geqslant y^1$ holds, it follows that $y^1 \in Y_E$, so that the proof is complete. □

From Theorem 3.1, if G is a face of Y and some point in $(ri\ G)$ belongs to Y_E, then *all* of $(ri\ G)$ belongs to Y_E *and* all points of $(rb\ G)$ contained in G belong to Y_E as well. Notice that these conclusions hold regardless of whether G is a closed set, an open set, or neither of these types of sets. Theorem 3.1 generalizes a result in [50] that applies only to multiple objective linear programs.

From Theorem 3.1, we have the following fundamental result for Y_E. The proof is an easy exercise.

COROLLARY 3.1. *Let T_E denote the collection of all relative interiors $(ri\ G)$ of faces G of Y that satisfy $(ri\ G) \subseteq Y_E$. Then T_E is a partition of Y_E, i.e., the sets in T_E are disjoint and their union is Y_E.*

Notice that Theorem 3.1 also implies that Y_E is a union of faces of Y, where the elements in this union need not be disjoint and are not necessarily open sets. Furthermore, since Y is itself a face of Y, Theorem 3.1 also implies the following result. (See Lemma 7.2 (i) in [48] for a related result).

COROLLARY 3.2. *If $(ri\ Y) \cap Y_E \neq \emptyset$, then every point in Y belongs to Y_E. Otherwise, $Y_E \subseteq (rb\,Y)$.*

Proof. If $(ri\ Y) \cap Y_E \neq \emptyset$, then, by setting G equal to Y in Theorem 3.1, we conclude that $Y \subseteq Y_E$. If $(ri\ Y) \cap Y_E = \emptyset$, then, by definition, $Y_E \subseteq (rb\,Y)$ must hold. □

The following are counterparts in decision space to Theorem 3.1 and its two corollaries. These results can be proven by using arguments analogous to those used to show the three outcome space results.

THEOREM 3.2. *Let F be a face of X. If $(ri\ F)$ contains a point $x^\circ \in X_E$, then $F \subseteq X_E$.*

COROLLARY 3.3. *Let S_E denote the collection of all relative interiors $(ri\ F)$ of faces F of X that satisfy $(ri\ F) \subseteq X_E$. Then S_E is a partition of X_E, i.e., the sets in S_E are disjoint and their union is X_E.*

COROLLARY 3.4. *If $(ri\ X) \cap X_E \neq \emptyset$, then every point in X belongs to X_E. Otherwise, $X_E \subseteq (rb\ X)$.*

For the special case when problem (MLC) is a multiple objective linear program, i.e., the convex set X is polyhedral, it has been observed empirically that one can typically expect to encounter fewer efficient extreme points and faces in Y_E than in X_E (cf., e.g. [12, 15, 20, 22]). Moreover, the dimensions of efficient faces are generally much smaller in Y_E than in X_E. The latter has been attributed, at least in part, to the fact that the dimension p of the outcome space is generally smaller than (and often *much* smaller than) the dimension n of the decision space [12]. In the remainder of this section, we shall formally analyze the geometrical issues raised by these observations, but in the more general setting of the multiple objective convex program (MLC).

THEOREM 3.3. *Suppose that in addition to being nonempty and convex, X is a compact set. Then for any $y^\circ \in Y_{\text{ex}}$, there exists a point $x^\circ \in X_{\text{ex}}$ such that $Dx^\circ = y^\circ$.*

Proof. Assume that $y^\circ \in Y_{\text{ex}}$. Suppose, to the contrary, that if $Dx = y^\circ$ for some $x \in X$, then $x \notin X_{\text{ex}}$. Choose any point $\overline{x}$ in X such that $D\overline{x} = y^\circ$. Then, by assumption, $\overline{x} \notin X_{\text{ex}}$. Since X is a compact, convex set, this implies that there exist points $x^1, x^2, \ldots, x^q \in X_{\text{ex}}$ and scalars $\alpha_1, \alpha_2, \ldots, \alpha_q > 0$ that sum to one such that

$$\overline{x} = \sum_{j=1}^{q} \alpha_j x^j,$$

where $q \geqslant 2$ [37]. Since $D\overline{x} = y^\circ$, this implies that

$$y^\circ = \sum_{j=1}^{q} \alpha_j y^j, \tag{3}$$

where, for each $j = 1, 2, \ldots, q, y^j = Dx^j \in Y$. For each $j = 1, 2, \ldots, q$, by assumption, since $x^j \in X_{\text{ex}}, Dx^j \neq y^\circ$ must hold. Therefore, $y^j \neq y^\circ, j = 1, 2, \ldots, q$.

Since $y^\circ \in Y_{\text{ex}}$, it is easy to see that $Y \backslash \{y^\circ\}$ is a convex set. For each $j = 1, 2, \ldots, q$, since $y^j \in Y$ and $y^j \neq y^\circ$, it follows that $y^j \in Y \backslash \{y^\circ\}$. Therefore, any convex combination of the points $y^j, j = 1, 2, \ldots, q$, must lie in $Y \backslash \{y^\circ\}$. But since Y is a convex set and the scalars $\alpha_j, j = 1, 2, \ldots, q$, are positive and sum to one, this contradicts (3), and the proof is complete. □

From Proposition 2.1 and Theorem 3.3, for any element y° of the efficient outcome set Y_E of problem (MLC) that is an extreme point of Y, there will exist an element x° of the efficient decision set X_E that is an extreme point of X such that $Dx^\circ = y^\circ$, provided that X is compact. This implies that when X is compact, the number of efficient extreme points of X is guaranteed to be at least as great as the number of efficient extreme points of Y. This verifies empirical observations that have been made for the special case of multiple objective linear programs with bounded decision sets.

REMARK 3.1. The point x° in Theorem 3.3 need not be unique. In fact, large numbers of extreme points x of X may be mapped by D into a single extreme point $y^\circ \in Y$, even in the case when X and Y are compact polyhedra. Furthermore, from Proposition 2.1, if $y^\circ \in Y_E$, then each of these extreme points x will belong to X_E. To illustrate these possibilities, consider the following simple example.

EXAMPLE 3.1. Let

$$X = \{x \in \mathbb{R}^n | 0 \leqslant x_j \leqslant 1, j = 1, 2, \ldots, n\},$$

and let D be the $2 \times n$ matrix whose first two columns form the 2×2 identity matrix and whose remaining columns contain all zeroes. Then Y is a unit square in $\mathbb{R}^2$, and $(y^\circ)^T = (1, 1)$ is an efficient extreme point of Y. In this case, any point $x \in X$ with $x_1 = x_2 = 1$ satisfies $Dx = y^\circ$. Notice, in particular, that 2^{n-2} efficient extreme points x of X satisfy $Dx = y^\circ$. If $n = 32$, for instance, then over one billion efficient extreme points of the feasible decision set X are mapped by D into the single extreme point y° of the efficient outcome set!

REMARK 3.2. Example 3.1 shows that impressively-large numbers of efficient extreme points of X may be mapped by D in problem (MLC) onto the same efficient extreme point of Y. In addition, similarly-large numbers of efficient extreme points of X may map into *non-extreme* efficient points of Y. The next example demonstrates this possibility.

EXAMPLE 3.2. Consider again Example 3.1, but with D replaced by the $2 \times n$ matrix whose entries in row one are all equal to 1.0 and in row two are all equal to -1.0. Then Y is the closed line segment in $\mathbb{R}^2$ connecting the origin and the point $(n, -n)$, and $Y_E = Y$. In this case, $(2^n - 2)$ of the 2^n efficient extreme points of X are mapped by D into non-extreme efficient points of Y. If $n = 30$, for example, this implies that over one billion efficient extreme points of X map into non-extreme efficient points of Y.

REMARK 3.3. The assumption that X is compact cannot be deleted from Theorem 3.3. This implies that when X is not compact, the efficient outcome set may contain *more* extreme points than the efficient decision set. This phenomenon can occur even in multiple objective linear programming. The next two examples illustrate these comments.

EXAMPLE 3.3. Let $C = \{x \in \mathbb{R}^3 | 0 \leqslant x_j \leqslant 1, j = 1, 2, 3\}$, and let X be the non-closed convex set obtained from C by deleting the eight extreme points of C. Define D to be the 2×3 matrix whose first two columns form the 2×2 identity matrix and whose last column is the zero vector in $\mathbb{R}^2$. Then Y contains four extreme points and one efficient extreme point, but X contains *no* extreme points whatsoever. In particular, none of the pre-images under D of the efficient extreme point $(y^\circ)^T = (1, 1)$ of Y is an extreme point of X, and Y_E contains more extreme points than X_E.

EXAMPLE 3.4. Let $X = \{x \in \mathbb{R}^3 | 0 \leqslant x_1, x_2 \leqslant 1\}$, and let D be defined as in Example 3.3. Then X is a nonempty, unbounded polyhedral set, and problem (MLC) is a multiple objective linear program. Notice that although the outcome set contains four extreme points and one efficient extreme point, the decision set X is devoid of extreme points. As in Example 3.3, then, in this example none of the pre-images under D of the efficient extreme points of Y is an extreme point of X, and Y_E contains more extreme points than X_E.

THEOREM 3.4. *Let G be an arbitrary face of Y. Then $F = \{x \in X | Dx \in G\}$ is a face of X, and* $\dim F \geqslant \dim G$.

Proof. To show that F is a face of X, we will use a proof by contradiction. Towards this end, suppose that F is not a face of X. By definition, since G is a face of Y, G is a convex set. Therefore $\{x \in \mathbb{R}^n | Dx \in G\}$ is also a convex set [37]. Since $F = X \cap \{x \in \mathbb{R}^n | Dx \in G\}$, and the intersection of two convex sets is again a convex set, F is a convex subset of X. This and the assumption that F is not a face of X imply, by definition of a face, that we may choose two points $u, v \in X$, at least one of which is not in F, such that there exists a point q in the relative interior of the closed line segment connecting u and v that satisfies $q \in F$.

Since q lies in the relative interior of the closed line segment connecting u and v, we may choose a scalar $\theta \in (0, 1)$ such that $q = \theta u + (1 - \theta)v$. It follows that $Dq = \theta Du + (1 - \theta)Dv$, which, by the fact that $q \in F$, implies that $\theta Du + (1 - \theta)Dv \in G$. Since G is a face of Y and the points Du and Dv belong to Y, this implies that both Du and Dv must belong to G. From this and the fact that both u and v belong to X, we conclude that both u and v belong to F. But this is untenable, since u and v were chosen so that at least one of them is not a member of F. This implies that F must be a face of X.

To complete the proof, we will show that it is impossible for $\dim G$ to exceed $\dim F$. Towards this end, suppose that $\dim G = k$. Then we may choose $k + 1$

affinely independent points $g^j \in G, j = 0, 1, \ldots, k$. If dim G were to exceed dim F, then every set of $k + 1$ points in F would necessarily be affinely dependent. In particular, any $k + 1$ points $f^j, j = 0, 1, \ldots, k$, in F satisfying $Df^j = g^j$ for each j would be affinely dependent. However, it is easy to see by using simple algebra that this would imply that the points $g^j, j = 0, 1, \ldots, k$, are not affinely independent, which is a contradiction. Thus, dim G cannot exceed dim F, and the proof is complete. □

REMARK 3.4. Theorem 3.4 states that given any face G in the outcome set Y of X under D, the set of all points in X that are mapped by D into G form a face F of X whose dimension is at least as great as the dimension of G. Unfortunately, however, the reverse is not true, i.e. not every face of X is mapped by D onto a face of Y. In particular, let F be a face of X that maps onto a face $G = D[F]$ of Y. Then there can be many faces of X that are strict subsets of F that are mapped by D onto convex subsets of G that intersect the relative interior of G but are *not* faces of Y. Furthermore, the dimensions of F and of these strict subsets of F can far exceed the dimension of G. Put another way, while some faces F of X map onto faces G of Y, there may be *much larger* numbers of faces $\overline{F}$ of X that are strict subsets of these types of faces that map into *non-facial* convex subsets of Y. Furthermore, the dimensions of the faces F and of these sorts of strict facial subsets $\overline{F}$ of F *can far exceed* the dimensions of the facial images G in Y of the faces F. These observations hold for efficient faces as well, even in the case where problem (MLC) is a multiple objective linear program.

To illustrate Remark 3.4, consider Example 3.2. Let G denote the closed line segment in $\mathbb{R}^2$ that connects the origin and the point $(n, -n)$. Then in this example, $G \subseteq Y_E$ and, since $G = Y, G$ is a face of Y. Notice that the dimension of G is one. By Proposition 2.1 and Theorem 3.4, the set $F = \{x \in X | Dx \in G\}$ is an efficient face of X of dimension one or more. In fact, in this case, F equals X itself, so that it has dimension n, which, of course, can far exceed one. Notice that every strict subface of F is efficient. Furthermore, except for the two efficient extreme point subfaces given by $\{(0, 0, \ldots, 0)^T\} \subseteq \mathbb{R}^n$ and $\{(1, 1, \ldots, 1)^T\} \subseteq \mathbb{R}^n$, every one of these efficient subfaces $\overline{F}$ maps into an efficient *non-facial* subset of $G = Y$. These non-facial subsets of G are convex sets. In particular, each one is either a single point in the relative interior of the line segment G or a closed line-segment subset of G other than G itself. Notice further that extremely-large numbers of efficient subfaces of the type $\overline{F}$ exist. For instance, there are $n2^{n-1}$ efficient edges of F alone that are subfaces of this type. Finally, observe that the dimensions of the efficient subfaces of F of this type vary from 0 to $(n - 1)$. For $n = 30$, for instance, these observations imply that although Y_E is a simple one-dimensional line segment in $\mathbb{R}^2$ consisting of just three efficient faces, X_E consists of billions of efficient faces of dimensions as large as 30, all but three of which are mapped under D into non-facial subsets of Y_E.

REMARK 3.5. Notice in the proof of Theorem 3.4 that to show $\dim F \geqslant \dim G$, only the convexity of F and of $G = D[F]$ were used. It follows that the dimension of the image of *any* convex set $S \subseteq \mathbb{R}^n$ under D is never greater than $\dim S$.

4. Complete Efficiency, Maximal Faces and Collapsing

Theorems 3.1 and 3.2 immediately imply the following result.

PROPOSITION 4.1. *Let F and G be arbitrary faces of X and of Y, respectively. Then $F \subseteq X_E$ if and only if $(ri\ F)$ contains a point $x^\circ \in X_E$, and $G \subseteq Y_E$ if and only if $(ri\ G)$ contains a point $y^\circ \in Y_E$.*

In a moment we shall use Proposition 4.1 to help find conditions under which the image of an efficient face of X under D is an efficient face of Y. But first let us study some more direct uses of the proposition.

DEFINITION 4.1. [3, 7]. The multiple objective convex program (MLC) is said to be *completely efficient* when $X = X_E$.

Although complete efficiency seems to be a relatively uncommon phenomenon, the issue of how frequently it occurs has yet to be addressed [3]. Notice from Definition 4.1 and Proposition 2.1 that problem (MLC) is also completely efficient if and only if $Y = Y_E$.

In problem (MLC), since X and Y are themselves faces of X and Y, respectively, Proposition 4.1 gives necessary and sufficient conditions not only for faces of X and of Y to be efficient, but also for problem (MLC) to be completely efficient. Therefore, Proposition 4.1 has a number of potential uses. It would thus be advantageous to use Proposition 4.1 to derive computational tests for the efficiency of a face of X or of Y. Various types of such tests can be envisioned. One such test is provided in the next result. This test relies on the solution of a convex programming problem similar to problems used in the past to test for the efficiency of individual points, rather than entire faces (cf., e.g., 4–5, 19, 21, 42 and references therein).

THEOREM 4.1. *Assume that G is a face of Y, and let y° be an arbitrary point in $(ri\ G)$. Then $G \subseteq Y_E$ if and only if the optimal value of the convex programming problem*

$$\sup\langle e, y\rangle - \langle e, y^\circ\rangle, \quad \text{s.t. } y \geqslant y^\circ, \quad y \in Y$$

equals 0, where $e \in \mathbb{R}^p$ denotes the vector of ones and $\langle\cdot,\cdot\rangle$ denotes the inner product.

Proof. The proof follows easily from Proposition 4.1. □

Notice, in particular, that if $G = Y$ in Theorem 4.1, then the theorem provides a test for the complete efficiency of problem (MLC). A result analogous to Theorem

4.1 can be easily derived which provides a test for the efficiency of an arbitrary face of X. Of course faces of X or Y need not be closed. In such cases, extra caution should be taken in using these face efficiency tests.

The concept of *maximal efficiency* has played a useful role in generating efficient faces in the decision sets of multiple objective linear programs [1, 18, 35, 48, 50]. The following definition extends this concept to the multiple objective convex program (MLC).

DEFINITION 4.2. A nonempty face F of set X (respectively, G of set Y) in problem (MLC) is called *maximally efficient* when it is efficient and no other efficient face $\overline{F}$ of X (resp., $\overline{G}$ of Y) exists such that $F \subseteq \overline{F}$ and $F \neq \overline{F}$ (resp., $G \subseteq \overline{G}$ and $G \neq \overline{G}$).

The next three results show that the notion of maximal efficiency also plays a key role in understanding the geometrical relationships between X_E and Y_E for both the multiple objective linear programming problem and problem (MLC).

THEOREM 4.2. *Suppose that F is a maximally efficient face of X. Then $G = \{y \in Y | y = Dx$ for some $x \in F\}$ is a maximally efficient face of Y.*

Proof. See Appendix.

Theorem 4.2 gives a sufficient condition for the image of an efficient face F of X under D to be an efficient face of Y. The condition is that F is maximally efficient in X. This condition, however, is not necessarily satisfied by every efficient face of X whose image under D is an efficient or maximally efficient face of Y. The following example demonstrates this.

EXAMPLE 4.1. Let D be defined as in Example 3.3, and let $X = \{x \in \mathbb{R}^3 | x_1^2 + x_2^2 \leqslant 1; x_1, x_2, x_3 \geqslant 0\}$. Then in problem (MLC), the image under D of the efficient face of X consisting of the extreme point $(x^\circ)^T = (1, 0, 0)$ is the efficient extreme point $(y^\circ)^T = (1, 0)$ of Y, which is a maximal efficient face of Y. Yet the efficient face $\overline{F} = \{x^\circ\}$ in X is not maximally efficient, since $\overline{F}$ is a strict subset of the efficient face F of X given by

$$F = \{x \in \mathbb{R}^3 | x = (1, 0, \alpha) \text{ for some } \alpha \geqslant 0\}. \tag{4}$$

THEOREM 4.3. *Suppose that G is a maximally efficient face of Y. Then $F = \{x \in X | Dx \in G\}$ is a maximally efficient face of X.*

Proof. From Proposition 2.1 and Theorem 3.4, F is an efficient face of X. Suppose, to the contrary, that F is not a maximally efficient face of X. Then for some other efficient face $\overline{F}$ of X, $F \subseteq \overline{F}$ and $F \neq \overline{F}$. We may assume without loss of generality that $\overline{F}$ is a maximally efficient face of X. From Theorem 4.2, this implies that $\overline{G} = \{y \in Y | y = Dx$ for some $x \in \overline{F}\}$ is an efficient face of Y. Since $F \subseteq \overline{F}$ and $F \neq \overline{F}$, we may choose a point $\overline{x} \in \overline{F} \backslash F$. Then, from the definitions of $\overline{G}$ and F, $D\overline{x} \in \overline{G} \backslash G$. Furthermore, since $F \subseteq \overline{F}$, these definitions also imply that $G \subseteq \overline{G}$. The latter two conclusions imply that $G \subseteq \overline{G}$ and $G \neq \overline{G}$. But since $\overline{G}$ is an efficient face of Y, these conclusions contradict that G is a maximally efficient face of Y, so that the proof is complete. □

From Theorems 4.2 and 4.3, we obtain the following result.

THEOREM 4.4. *The mapping D induces a one-to-one correspondence between the maximally efficient faces of the decision set X and the outcome set Y of problem (MLC).*

Proof. Let F be a maximally efficient face of X. Then, from Theorem 4.2, $G = \{y \in Y | y = Dx \text{ for some } x \in F\}$ is a corresponding maximally efficient face of Y. By applying Theorem 4.3 to G, we obtain that the set $\overline{F} = \{x \in X | Dx \in G\}$ is a maximally efficient face of X corresponding to G. Notice that $F \subseteq \overline{F}$. Since F is a maximally efficient face of X and $\overline{F}$ is an efficient face of X, this implies by Definition 4.2 that $F = \overline{F}$, so that the theorem is established. □

Dauer and Gallagher [14] have very recently shown Theorem 4.4 for the special case where X and Y are polyhedra, i.e., where problem (MLC) is a multiple objective linear program.

Theorems 4.2–4.4 establish that an organized relationship exists between the sets of faces in the efficient decision set and in the efficient outcome set of problem (MLC). In addition, they provide a structure for constructing algorithms to generate all or parts of these sets. For instance, they suggest that to construct algorithms for generating all of Y_E, it would be beneficial to view the task as one of generating all of the individual maximal faces of Y_E. By generating all of these faces, bookkeeping tasks could potentially be simplified, and individual maximal faces in X_E could be readily identified as pre-images under D of the maximal faces of Y_E.

For the special case of multiple objective linear programming with nonempty compact decision sets (i.e. for the case where X is a nonempty compact polyhedron), Dauer [12] found it useful to define the notion of "collapsing" to study the effects of the mapping D on faces of X_E. The definition of collapsing used by Dauer is as follows.

DEFINITION 4.3. [12]. Let X be a nonempty compact polyhedron. Then a face F of X is said to *collapse under D* when there exists a subface $\overline{F} \subseteq F$ of X, $\overline{F} \neq F$, such that $\dim D[\overline{F}] = \dim D[F]$.

The assumption in Definition 4.3 that X is a compact polyhedron is not essential to the concept of collapsing. Therefore, we will assume henceforth that Definition 4.3 is also valid when X is an arbitrary nonempty convex set in $\mathbb{R}^n$.

Notice from Definition 4.3 and [37] that when a face F of X collapses under D, there exists some strict subface $\overline{F}$ of F (which must necessarily satisfy $\dim \overline{F} < \dim F$) such that $D[\overline{F}] \subseteq D[F]$, but $\dim D[\overline{F}] = D[F]$. The collapsing of efficient faces in problem (MLC) is quite common, even when problem (MLC) is a multiple objective linear program [12]. Furthermore, maximally efficient faces in X_E are not immune to collapsing. As an illustration of the latter point, notice that the face F defined by (4) is a maximally efficient face of X in Example 4.1, but since $\overline{F} = \{(1,0,0)\}$ is a strict (efficient) subface of F for which $\dim D(\overline{F}) = \dim D(F) = 0$, F collapses under D in this example.

We will identify some of the types of faces of X (and hence of X_E) that are guaranteed *not* to collapse under D. Towards this end, consider the following definition.

DEFINITION 4.4. A nonempty face F of X in problem (MLC) is said to be *algebraically nondegenerate* with respect to D when

$$Dx = 0, x \in (\text{sub } F)$$

imply that $x = 0$.

Definition 4.4 generalizes a concept introduced for multiple objective linear programs by Philip [35]. Notice that if F is algebraically nondegenerate with respect to D, then so is any nonempty strict subface $\overline{F} \subseteq F$ of X. The next result generalizes a result of Dauer [12].

THEOREM 4.5. *An algebraically nondegenerate face of X with respect to D cannot collapse under D.*

Proof. See Appendix.

Since algebraically nondegenerate faces cannot collapse, yet collapsing is often observed empirically, one tentatively concludes that in practice, only a minority of faces is algebraically nondegenerate. Some additional theoretical justification for this tentative conclusion is provided by the next result (cf. [12] for the linear case).

THEOREM 4.6. *A nonempty face F of X is algebraically nondegenerate with respect to D if and only if $\dim D[F] = \dim F$.*

Proof. See Appendix.

REMARK 4.1. Although algebraic nondegeneracy is a sufficient condition for guaranteeing that collapsing does not occur, it is not a necessary one. To demonstrate this, consider the following example.

EXAMPLE 4.2. Let D be defined as in Example 3.3, and let $X = \{x \in \mathbb{R}^3 | x_1^2 + x_2^2 \leqslant 1, 0 < x_3 < 2\}$. Then the set X_E consists of a union of an infinite number of one-dimensional maximal efficient faces, each of the form

$$F_\theta = \{x \in \mathbb{R}^3 | x_1 = \theta, x_2 = (1-\theta^2)^{1/2}, x_3 \in (0,2)\},$$

where $\theta \in [0,1]$. All of the faces $F_\theta, \theta \in [0,1]$, are algebraically degenerate with respect to D. This is because for each $\theta \in [0,1], \overline{x} = (0,0,1)$, for instance, belongs to (sub F_θ) and satisfies $D\overline{x} = 0$. Yet none of these faces collapses under D, because none contains a strict subset that is a face of X.

5. Some Conclusions and Suggestions for Further Research

This article has presented a geometrical analysis of the efficient outcome set Y_E of problem (MLC). The analysis was motivated to a great extent by a desire to provide some foundations for developing algorithms for generating all or parts of Y_E. The results seem to indicate that, as in the case of multiple objective linear programming, great computational benefits can be expected to accrue if multiple objective algorithms for problem (MLC) were to work directly with the outcome set Y to find points and faces of Y_E rather than with the decision set X. Furthermore, the results also seem to indicate that such algorithms will benefit computationally by focusing on the generation of maximal efficient faces of Y_E. Given such faces, corresponding maximal efficient faces in the decision set X can then be identified. A potentially-fruitful topic for further research, then, is the development of algorithms for problem (MLC) of this type.

As mentioned in Section 1, by focusing on $Y^{\leqslant}$ and on $Y_E^{\leqslant}$, many (but not all) of the results given in the article for problem (MLC) can be extended to the more general multiple objective programming problem (MCC) in which the criteria are arbitrary concave functions on X. Another topic for further study, then, is the geometrical structure of the efficient decision and outcome sets of problem (MCC).

A third suggestion for further research is to study in more detail the effects that the linear mapping D has on an efficient face F of X in problem (MLC). In particular, given dim F, it would be of interest to have a means of precisely calculating dim $D[F]$. In addition, it would be valuable to be able to characterize when F will and will not collapse under D. Such knowledge could be valuable in computational contexts. For instance, prior knowledge that dim $D[F]$ is relatively small for certain efficient faces F of X could indicate to a decision maker that it is not worthwhile to explicitly generate either the face F or its image $D[F]$ in Y_E.

6. Appendix

Proof of Theorem 4.2. From Proposition 2.1, $G \subseteq Y_E$. To show that G is a face of Y, assume, to the contrary, that it is not. Notice that since F is a nonempty convex subset of $X, G = D[F]$ is a nonempty convex subset of Y [37]. Since X is also a nonempty convex set, this implies that the set $(XG)^{-1}$ defined by

$$(XG)^{-1} = \{x \in X | Dx \in G\}$$

is a nonempty convex set in $\mathbb{R}^n$ [37]. Also, from Proposition 2.1, $(XG)^{-1} \subseteq X_E$.

Since G is not a face of Y, we may choose points $y^1, y^2 \in Y$ such that either $y^1 \notin G$ or $y^2 \notin G$ or both, yet there exists a point g in the relative interior of the closed line segment L connecting y^1 and y^2 such that $g \in G$. Then, by the definitions of Y and of G, for some $x^1, x^2 \in X$ such that either $x^1 \notin F$ or $x^2 \notin F$ or both, $y^j = Dx^j$ for each $j = 1, 2$. Since $g \in G \cap (ri\ L)$, we may choose a point $x \in F$ and a scalar $\theta \in (0, 1)$ such that

$$g = Dx = \theta y^1 + (1 - \theta)y^2.$$

Substituting for y^1 and y^2 in the above, we obtain

$$\begin{aligned} g = Dx &= \theta Dx^1 + (1 - \theta)Dx^2 \\ &= D[\theta x^1 + (1 - \theta)x^2] \\ &= D\overline{x}, \end{aligned}$$

where $\overline{x} = \theta x^1 + (1 - \theta)x^2$. Since X is a convex set, $\overline{x} \in X$. However, since F is a face of X and x^1 and x^2 do not both belong to F, the definition of $\overline{x}$ implies that $\overline{x} \notin F$. Furthermore, since $\overline{x} \in X$ and $D\overline{x} = g \in G$, by definition of $(XG)^{-1}, \overline{x} \in (XG)^{-1}$. Notice that $F \subseteq (XG)^{-1}$. Since $\overline{x} \in (XG)^{-1}$, but $\overline{x} \notin F$, it follows that $F \neq (XG)^{-1}$. By Definition 4.2, since F is a maximally efficient face of X and $(XG)^{-1}$ is a nonempty convex set in X_E that contains F as a strict subset, $(XG)^{-1}$ cannot be a face of X.

From [37], since $(XG)^{-1}$ is a nonempty convex set, it has a nonempty relative interior $(ri(XG)^{-1})$. Furthermore, for [37], $(ri(XG)^{-1})$ is a relatively open convex set. Since $(ri(XG)^{-1})$ is also a subset of X, this implies by Theorem 18.2 in [37] that $(ri(XG)^{-1}) \subseteq (ri\ \overline{F})$ for some nonempty face $\overline{F}$ of X. Because $(ri(XG)^{-1}) \neq \emptyset$, this implies that $(ri(XG)^{-1}) \cap \overline{F} \neq \emptyset$. From Theorem 18.1 in [37], since $\overline{F}$ is a face of the convex set X and $(XG)^{-1}$ is a convex set in X, it follows that $(XG)^{-1} \subseteq \overline{F}$. This implies that $\overline{F}$ strictly contains F, since F is a strict subset of $(XG)^{-1}$. Furthermore, since $(XG)^{-1} \subseteq X_E$ and $(ri(XG)^{-1})$ is a nonempty subset of the relative interior of the face $\overline{F}$ of X, Proposition 4.1 implies that $\overline{F}$ is an efficient face of X. By Definition 4.2, however, this contradicts that F is a maximally efficient face of X, thus proving that G is a face of Y.

To finish the proof, we must show that the efficient face G of Y is maximally efficient in Y. Suppose, to the contrary, that there exists some efficient face $\hat{G}$ of Y such that $G \subseteq \hat{G}$ and $G \neq \hat{G}$. Then, from Proposition 2.1 and Theorem 3.4, the set $\hat{F}$ given by

$$\hat{F} = \{x \in X | Dx \in \hat{G}\}$$

is an efficient face of X. Notice that if $\hat{x} \in F$, then $\hat{x} \in X$ and $D\hat{x} \in G \subseteq \hat{G}$. This implies that $F \subseteq \hat{F}$. Furthermore, $F \neq \hat{F}$ must hold, since if $F = \hat{F}$ were true, then any $x \in X$ for which $Dx \in \hat{G}$ would satisfy $x \in F$ and $Dx \in G$, which would imply that $\hat{G} \subseteq G$. This, together with $G \subseteq \hat{G}$, would imply that $G = \hat{G}$, a contradiction to $G \neq \hat{G}$.

To recap, we have shown that $\hat{F}$ is an efficient face of X that satisfies $F \subseteq \hat{F}$ and $F \neq \hat{F}$. This, however, contradicts that F is a maximally efficient face of X, so that G is maximally efficient in Y. □

Proof of Theorem 4.5. Let F be an algebraically nondegenerate face of X with respect to D. We will show that this implies $\dim F = \dim D[F]$. Towards this end, let $q = \dim F$.

First, notice from Remark 3.5 that $\dim D[F] \leqslant q$. We need only show, then, that $\dim D[F] \geqslant q$.

We will show that because F is an algebraically nondegenerate face of X with respect to D, $\dim D[F] \geqslant q$ must hold. Towards this end, notice that since $\dim F = q$, we may choose $(q+1)$ affinely independent vectors $x^j, j = 1, 2, \ldots, q+1$, from F.

Consider the set A of q vectors in $\mathbb{R}^p$ given by

$$A = \{D(x^j - x^1) | j = 2, 3, \ldots, q+1\}.$$

Suppose that $\sum_{j=2}^{q+1} \alpha_j D(x^j - x^1) = 0$ for some scalars $\alpha_j, j = 2, 3, \ldots, q+1$. If we let $\hat{x} = 1(x^1 - x^1) + \sum_{j=2}^{q+1} \alpha_j (x^j - x^1)$, then this implies that $D\hat{x} = 0$. Notice that $\hat{x}$ may also be written

$$\hat{x} = \left[\left(1 - \sum_{j=2}^{q+1} \alpha_j \right) x^1 + \sum_{j=2}^{q+1} \alpha_j x^j \right] - x^1,$$

which is an element of the set H given by

$$H = (\operatorname{aff} F) - \{x^1\}.$$

Since $H = (\operatorname{sub} F)$ [37] and $D\hat{x} = 0$, this implies by the algebraic nondegeneracy of F that $\hat{x} = 0$. By definition of $\hat{x}$, this means that

$$\sum_{j=2}^{q+1} \alpha_j (x^j - x^1) = 0. \tag{5}$$

The affine independence of the set of vectors $\{x^j | j = 1, 2, \ldots, q+1\}$ implies that the vectors $(x^j - x^1), j = 2, 3, \ldots, q+1$, are linearly independent. From (5), this implies by definition that $\alpha_j = 0, j = 2, 3, \ldots, q+1$. It follows that set A is a set of linearly independent vectors. Therefore, the $(q+1)$ vectors $Dx^j, j = 1, 2, \ldots, q+1$, in $D[F]$ are affinely independent. By definition, this implies that $\dim D[F] \geqslant q$.

The arguments so far show that $\dim D[F] = \dim F$. From Definition 4.3 and the fact that any nonempty, strict subface of F must, like F, be algebraically nondegenerate with respect to D, this implies that F cannot collapse under D. □

Proof of Theorem 4.6. The *necessity* portion of this result follows from the proof of Theorem 4.5. To show the *sufficiency* portion of the result, we will prove the contrapositive.

Assume that F is a nonempty, algebraically degenerate face of X with respect to D. Let $\dim F = q$. From Remark 3.5, either $\dim D[F] = \dim F$ or $\dim D[F] < \dim F$. Showing that $\dim D[F] \neq \dim F$, then, is equivalent to showing that $\dim D[F] < \dim F$. To show the latter, we will show that for any set of $(q+1)$ points in F, their images under D form a set of $(q+1)$ affinely dependent points in $D[F]$. Since for *any* set Q of $(q+1)$ points in $D[F]$, there must exist $(q+1)$ points in F whose set of images under D precisely equal Q, this will imply the desired result, and the theorem will be proven.

Therefore, let us assume that $T = \{x^j | j = 1, 2, \ldots, q+1\}$ is a set of $(q+1)$ points in F. Then T is either an affinely dependent or an affinely independent set of $(q+1)$ vectors. If T is an affinely dependent set, then the definition of affine dependence and simple linear algebra can be easily used to show that $D[T]$ is an affinely dependent set of $(q+1)$ points in $D[F]$. Therefore, let us assume henceforth that T is a set of $(q+1)$ affinely independent points.

Because F is algebraically degenerate with respect to D, we may choose a point $\overline{x} \in \mathbb{R}^n$ such that

$$\overline{x} \neq 0, \tag{6}$$

$$\overline{x} \in (\text{sub } F), \tag{7}$$

and

$$D\overline{x} = 0. \tag{8}$$

Since $\dim F = q$ and $x^j, j = 1, 2, \ldots, q+1$, are affinely independent vectors in F, any point in (aff F) can be expressed as an affine combination of the vectors $x^j, j = 1, 2, \ldots, q+1$. From [37], since $x^1 \in$ (aff F), this implies that any point in (sub F) can be expressed as the algebraic difference between some affine combination of the points $x^j, j = 1, 2, \ldots, q+1$, and x^1. In particular, from (7), this implies that for some scalars $\alpha_j, j = 1, 2, \ldots, q+1$, whose sum is one,

$$\overline{x} = \left(\sum_{j=1}^{q+1} \alpha_j x^j\right) - x^1.$$

By rearranging the right-hand side of this equation and using the fact that the scalars $\alpha_j, j = 1, 2, \ldots, q+1$, sum to one, we obtain

$$\overline{x} = \sum_{j=2}^{q+1} \alpha_j (x^j - x^1).$$

This, from (6) and (8), implies that

$$\sum_{j=2}^{q+1} \alpha_j D(x^j - x^1) = 0 \tag{9}$$

and

$$\sum_{j=2}^{q+1} \alpha_j (x^j - x^1) \neq 0.$$

From the latter inequality, it follows that $\alpha_j, j = 2, 3, \ldots, q+1$, cannot all equal zero. From this and (9) it follows that the vectors $(Dx^j - Dx^1), j = 2, 3, \ldots, q+1$, are linearly dependent. Therefore, $\{Dx^j | j = 1, 2, \ldots, q+1\} = D[T]$ is an affinely dependent set of $(q+1)$ vectors in $D[F]$, and the proof is complete. □

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